

Quotients of Primes in a Quadratic Number Ring

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Abstract

It has been established on many occasions that the set of quotients of prime numbers is dense in the set of positive real numbers. More recently, it has been proved in the *Monthly* that the set of quotients of primes in the Gaussian integers is dense in the complex plane. In this article, we not only extend this result to any imaginary quadratic number ring, but also prove that the set of quotients of primes in any real quadratic number ring is dense in the set of real numbers.

1 Introduction

It is a standard fact from Real Analysis that \mathbb{Q} is a dense subset of \mathbb{R} . However, it is not as well-known that the set of quotients of *prime* numbers is a dense subset of $\mathbb{R}_{>0}$. One of the earliest appearances of this fact is in Sierpiński's textbook on Number Theory [12]. This fact, along with a few variations of it, has been studied and explored in great detail in the *Monthly* and elsewhere through the years; see [1], [4], [3], [10], and [13] for instance. These results are proved by using some variant of the Prime Number Theorem, which as a reminder we now state in its classic formulation below.

Theorem 1.1 *If $\pi(x)$ denotes the number of prime numbers in \mathbb{N} less than or equal to x , then $\pi(x) \sim \frac{x}{\ln x}$. In other words, $\lim_{x \rightarrow \infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1$.*

Recently, Garcia established in the *Monthly* [5] that the set of quotients of primes of *Gaussian integers* is dense in \mathbb{C} by using a generalization of the Prime Number Theorem to Gaussian integers (which is stated in the next section). Inspired by Garcia's work, we first establish his density result for the Gaussian integers in a slightly different manner. This derivation not only allows us to establish the generalization that the set of quotients of primes belonging to a fixed congruence class of Gaussian integers is dense in \mathbb{C} , but also (with an appropriately modified variant of the Prime Number Theorem) to show that the set of quotients of primes from any fixed imaginary quadratic ring is dense in \mathbb{C} .

After accomplishing this task, we turn our attention to real quadratic number rings, where we show that the set of quotients of primes from a given real quadratic ring is dense in \mathbb{R} .

In this article, the prime numbers in \mathbb{N} will be referred to as **rational primes**. This will allow us to distinguish them from the primes in a given quadratic number ring.

2 Quotients of Gaussian Primes Revisited

Recall that the ring of **Gaussian integers** $\mathbb{Z}[i]$ is the set of integers in the field $\mathbb{Q}(i)$ and have the form $a + bi$ where $a, b \in \mathbb{Z}$. This set is well-known to be a UFD with units $\pm 1, \pm i$. It can be shown (see [9] or [14]) that a nonzero Gaussian integer is a prime if and only if it is an associate of p (namely $\pm p$ and $\pm pi$) for some rational prime $p \equiv 3 \pmod{4}$, or it is of the form $a + bi \in \mathbb{Z}[i]$ where $a^2 + b^2$ is a rational prime. We will refer to these primes as **Gaussian primes**.

To establish the density of Gaussian primes in the complex plane, we use the following variant of the prime number theorem given by Kubilius [6].

Proposition 2.1 *Fix $0 \leq \theta_1 < \theta_2 \leq 2\pi$, and let $\Pi(x; \theta_1, \theta_2)$ denote the number of Gaussian primes ρ satisfying $\theta_1 \leq \arg \rho \leq \theta_2$ and $|\rho|^2 \leq x$. Then,*

$$\Pi(x; \theta_1, \theta_2) \sim \frac{4(\theta_2 - \theta_1)}{2\pi} \cdot \frac{x}{\ln x}.$$

Not only does this proposition imply that there are infinitely many Gaussian primes in any given sector, but also that the Gaussian primes are angularly equidistributed. Moreover, by looking at a sector strictly contained inside the sector $[\theta_1, \theta_2]$, we find that there are infinitely many Gaussian primes in the *interior* of the sector $[\theta_1, \theta_2]$.

One further consequence of this proposition which we will use is that there are infinitely many Gaussian primes in an annular sector of the form

$$\{z \in \mathbb{C} : ax \leq |z|^2 \leq bx \text{ and } \theta_1 \leq \arg z \leq \theta_2\}$$

where $x > 0$ is sufficiently large. This follows from straightforward calculus:

$$\begin{aligned} \lim_{x \rightarrow \infty} [\Pi(bx; \theta_1, \theta_2) - \Pi(ax; \theta_1, \theta_2)] &= \lim_{x \rightarrow \infty} \Pi(bx; \theta_1, \theta_2) \left[1 - \frac{\Pi(ax; \theta_1, \theta_2)}{\Pi(bx; \theta_1, \theta_2)} \right] \\ &= \lim_{x \rightarrow \infty} \Pi(bx; \theta_1, \theta_2) \left[1 - \frac{ax \ln(bx)}{bx \ln(ax)} \right] \\ &= \left(1 - \frac{a}{b} \right) \lim_{x \rightarrow \infty} \Pi(bx; \theta_1, \theta_2) \\ &= \infty. \end{aligned}$$

We now are in a position to state and prove the main theorem of this section.

Theorem 2.2 *The set of quotients of Gaussian primes is dense in \mathbb{C} .*

Proof: In order to prove this, it suffices to show that any open annular sector

$$\{z \in \mathbb{C} : \psi_1 < \arg z < \psi_2, 0 < r < |z| < R\}$$

contains a quotient of Gaussian primes. Moreover, since associates of primes are again primes, we can assume without loss of generality that $\psi_1, \psi_2 \in [0, \frac{\pi}{2}]$.

From our previous remarks, we know that for any $\theta \in (0, 2\pi]$, we have

$$\lim_{x \rightarrow \infty} \left[\Pi\left(\frac{x}{r^2}; 0, \theta\right) - \Pi\left(\frac{x}{R^2}; 0, \theta\right) \right] = \infty.$$

Moreover, Proposition 2.1 implies that there are infinitely many Gaussian primes in the sector (ψ_1, ψ_2) of arbitrarily large magnitude. Therefore, there exists a Gaussian prime π_1 in the sector (ψ_1, ψ_2) with sufficiently large magnitude such that

$$\Pi\left(\frac{|\pi_1|^2}{r^2}; 0, \xi\right) - \Pi\left(\frac{|\pi_1|^2}{R^2}; 0, \xi\right) \geq 2,$$

where $\xi = \min\{\psi_2 - \arg(\pi_1), \arg(\pi_1) - \psi_1\}$.

Next, the inequality in the last assertion implies that there exists a Gaussian prime π_2 such that $\frac{|\pi_1|}{R} < |\pi_2| < \frac{|\pi_1|}{r}$ and $0 < \arg(\pi_2) < \xi$.

We are almost done. The first of the last two inequalities immediately yields $r < \left|\frac{\pi_1}{\pi_2}\right| < R$. The second of the last two inequalities, along with the fact that $\arg\left(\frac{\pi_1}{\pi_2}\right) = \arg(\pi_1) - \arg(\pi_2)$, implies that $\psi_1 < \arg\left(\frac{\pi_1}{\pi_2}\right) < \psi_2$ as required. \square

It should be remarked that this proof does not rely on a specific type of a Gaussian prime, while the proof in [5] fixes the Gaussian prime in the denominator be a rational prime congruent to 3 mod 4. Although this latter idea simplifies the density proof a bit with regards to the argument of the prime, it does not allow for the denominator to be an associate of a Gaussian prime that is not a rational prime.

This distinction will be exploited to show how flexible our density proof is, as we can easily modify it to give a Gaussian integer analogue of the following fact of Micholson [10] and Sarni [13]:

For any $a, m \in \mathbb{N}$ such that $\gcd(a, m) = 1$, then the set of quotients of primes congruent to a mod m is dense in $\mathbb{R}_{>0}$.

To state and prove this generalization, we note that the Gaussian integers have a notion of congruence as well. Fix $\gamma \in \mathbb{Z}[i]$; for $\alpha, \beta \in \mathbb{Z}[i]$ we write $\alpha \equiv \beta \pmod{\gamma}$ iff $\gamma \mid (\beta - \alpha)$. With this notation, we can now state the following ‘Dirichlet’ variant of the prime number theorem by Kubilius (in [6]).

Proposition 2.3 Fix $0 \leq \theta_1 < \theta_2 \leq 2\pi$, and let $\Pi(x; \theta_1, \theta_2; \beta, \gamma)$ denote the number of Gaussian primes $\rho \equiv \beta \pmod{\gamma}$ satisfying $\theta_1 \leq \arg \rho \leq \theta_2$ and $|\rho|^2 \leq x$. Then,

$$\Pi(x; \theta_1, \theta_2; \beta, \gamma) \sim \frac{4(\theta_2 - \theta_1)}{2\pi\phi(\gamma)} \cdot \frac{x}{\ln x},$$

where $\phi(\gamma)$ denotes the number of invertible congruence classes modulo γ .

It should be noted that this proposition asserts that the Gaussian primes are equidistributed both by argument and by congruence class. Moreover, the ϕ function above is an extension of the Euler phi function to $\mathbb{Z}[i]$.

By instead using this variation of the Gaussian prime number theorem in the previous density proof, we immediately obtain the following result.

Theorem 2.4 Fix $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}[i]$ such that $\gcd(\beta_1, \gamma_1) = \gcd(\beta_2, \gamma_2) = 1$. Then, the set of quotients

$$\left\{ \frac{\pi_1}{\pi_2} : \pi_1 \equiv \beta_1 \pmod{\gamma_1} \text{ and } \pi_2 \equiv \beta_2 \pmod{\gamma_2} \text{ are Gaussian primes} \right\}$$

is dense in \mathbb{C} .

Proof: As before, it suffices to show that for fixed $0 \leq \psi_1, \psi_2 \leq \frac{\pi}{2}$,

$$\{z \in \mathbb{C} : 0 \leq \psi_1 < \arg z < \psi_2 \leq \frac{\pi}{2}, 0 < r < |z| < R\}$$

contains a quotient of Gaussian primes of the form specified in the theorem.

For any $\theta \in (0, 2\pi]$, it follows as before (this time using Proposition 2.3 with Calculus) that

$$\lim_{x \rightarrow \infty} \left[\Pi\left(\frac{x}{r^2}; 0, \theta; \beta_2, \gamma_2\right) - \Pi\left(\frac{x}{R^2}; 0, \theta; \beta_2, \gamma_2\right) \right] = \infty.$$

Moreover, Proposition 2.3 implies that there are infinitely many Gaussian primes congruent to $\beta_1 \pmod{\gamma_1}$ in the sector (ψ_1, ψ_2) . Therefore, there exists a Gaussian prime $\pi_1 \equiv \beta_1 \pmod{\gamma_1}$ in the sector (ψ_1, ψ_2) with sufficiently large magnitude such that

$$\Pi\left(\frac{|\pi_1|^2}{r^2}; 0, \xi; \beta_2, \gamma_2\right) - \Pi\left(\frac{|\pi_1|^2}{R^2}; 0, \xi; \beta_2, \gamma_2\right) \geq 2,$$

where $\xi = \min\{\psi_2 - \arg(\pi_1), \arg(\pi_1) - \psi_1\}$.

Next, the inequality in the last assertion implies that there exists a Gaussian prime $\pi_2 \equiv \beta_2 \pmod{\gamma_2}$ such that $\frac{|\pi_1|}{R} < |\pi_2| < \frac{|\pi_1|}{r}$ and $0 < \arg(\pi_2) < \xi$.

As in the proof of Theorem 2.2, it now follows that $r < \left| \frac{\pi_1}{\pi_2} \right| < R$ and $\psi_1 < \arg\left(\frac{\pi_1}{\pi_2}\right) < \psi_2$. □

3 Background on Quadratic Number Rings

Having established that the set of quotients of Gaussian primes is dense in the complex plane, we consider the next natural question: Does this result extend to any quadratic number ring? Before we proceed to answer this question, we first fix some notation for a quadratic number ring.

A **quadratic number ring** is the ring of integers \mathcal{O} in the quadratic field $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer. In fact, $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ if $d \not\equiv 1 \pmod{4}$, and $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1 \pmod{4}$; for proofs of this (or any facts in this section), check any textbook on Algebraic Number Theory, such as [9] and [14]. We call the quadratic number ring *real* or *imaginary*, according to d being positive or negative, respectively. Note that the ring of Gaussian integers are a special case of an imaginary quadratic number ring with $d = -1$.

In addition to the basic field operations, a quadratic field has an additional operation of **conjugation** which generalizes complex conjugation. For any $\alpha = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$, we define its conjugate to be $\alpha' = a - b\sqrt{d}$. It is easy to check that for any $\alpha, \beta \in \mathbb{Q}(\sqrt{d})$, we have $(\alpha + \beta)' = \alpha' + \beta'$ and $(\alpha\beta)' = \alpha' \cdot \beta'$.

Armed with this operation, we define the **norm** of $\alpha \in \mathbb{Q}(\sqrt{d})$ as $N(\alpha) = \alpha\alpha'$. This is always a rational number, and moreover is an integer when $\alpha \in \mathcal{O}$. Observe that in the case where $d < 0$, it should be noted that $N(\alpha) = |\alpha|^2$.

Unlike \mathbb{Z} , it turns out that \mathcal{O} is not generally a UFD. (For that matter, only *nine* imaginary quadratic number rings are UFDs.) To indicate what can happen, as with commutative rings, we say that π is a **prime** element of \mathcal{O} if π is nonzero and a nonunit such that whenever $\pi \mid \alpha\beta$ for some $\alpha, \beta \in \mathcal{O}$, then $\pi \mid \alpha$ or $\pi \mid \beta$. This is to be distinguished from a nonzero non-unit element $\gamma \in \mathcal{O}$ whose only divisors are units in \mathcal{O} or associates of γ , which is called an **irreducible**. Prime elements are always irreducible, and conversely only in a UFD such as \mathbb{Z} or $\mathbb{Z}[i]$ are irreducibles always primes. To get around this problem, we consider ideals in \mathcal{O} . It is standard fact that \mathcal{O} has unique factorization into prime *ideals*.

One way to ‘measure’ how far \mathcal{O} is from being a UFD is as follows: we set up an equivalence relation on the ideals in \mathcal{O} as follows: we say that two ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O} are equivalent if there exist nonzero $\alpha, \beta \in \mathcal{O}$ such that $\langle \alpha \rangle \mathfrak{a} = \langle \beta \rangle \mathfrak{b}$. This equivalence partitions the ideals into disjoint ideal classes which form a finite abelian group called the **class group** of \mathcal{O} . Its cardinality is called the **class number** of \mathcal{O} and is denoted by h . For what we need, it should be noted that the trivial ideal class is the class of *principal* ideals, and \mathcal{O} is a UFD iff $h = 1$ (as all ideals in a UFD are principal). The trivial ideal class is of chief interest to us, because a generator of a principal prime ideal is a prime element.

4 Quotients of Primes in an Imaginary Quadratic Ring

In this section, we extend the density result for the Gaussian integers to any imaginary quadratic number ring. To do this, we use the following prime number theorem of Kubilius [7] (which can also be found as early without a growth estimate in the seminal work of Hecke [2]).

Proposition 4.1 *Fix $0 \leq \theta_1 < \theta_2 \leq 2\pi$, and let $\Pi(x; \theta_1, \theta_2)$ denote the number of prime elements $\rho \in \mathcal{O}$ with $N(\rho) \leq x$ and $\theta_1 \leq \arg \rho \leq \theta_2$. Then,*

$$\Pi(x; \theta_1, \theta_2) \sim \frac{g(\theta_2 - \theta_1)}{2\pi h} \cdot \frac{x}{\ln x},$$

where g denotes the (finite) number of units in \mathcal{O} .

The presence of h in this prime number formula signifies that we are looking at the ideal class of principal ideals. In turn, the presence of the factor g follows from the fact that a generator of a principal ideal is unique up to multiplication by a unit in \mathcal{O} . As a sanity check, observe that when $d = -1$, we have $\mathcal{O} = \mathbb{Z}[i]$. Consequently, $g = 4$, $h = 1$, and Proposition 4.1 reduces to Proposition 2.1 as expected.

Now, we have the desired variant of a prime number theorem to deduce the desired density result for any imaginary quadratic number ring.

Theorem 4.2 *The set of quotients of primes in an imaginary quadratic ring \mathcal{O} is dense in the complex plane.*

Proof: In order to prove this, it suffices to show that any annular sector

$$\{z \in \mathbb{C} : \psi_1 < \arg z < \psi_2, 0 < r < |z| < R\}$$

contains a quotient of primes in \mathcal{O} . Moreover, since associates of primes are again primes, we can assume without loss of generality that $\psi_1, \psi_2 \in [0, \frac{2\pi}{g}]$. Now, adapt the proof of Theorem 2.2, but using Proposition 4.1 instead of Proposition 2.1. \square

5 Quotients of Primes in a Real Quadratic Number Ring

We now turn our attention to real quadratic number rings. These rings are quite different than their imaginary counterparts. First, since a real quadratic number ring is a subset of \mathbb{R} , we will be stating a density statement in \mathbb{R} instead of \mathbb{C} . Secondly, a real quadratic number ring has *infinitely* many units. Fortunately, any such unit can be written in the form $\pm \eta^k$ for some integer k and fixed unit

$\eta \in \mathcal{O}$. We call $\eta \in \mathcal{O}$ a **fundamental unit** of \mathcal{O} . Noting that whenever η is a fundamental unit, so are $-\eta$ and $\pm\eta^{-1}$, we can always fix a fundamental unit that is greater than 1.

Since any prime number in \mathcal{O} has infinitely many associates, any prime number theorem concerning such primes has to address this obstacle. Such a result also dates back to the work of Hecke [2], but we will use the following refinement of this by Rademacher [11]. (Somewhat surprisingly, these results have not been quoted much, leave alone in English, since their publications.) To state the theorem, we need the following definition: we say that $\alpha \in \mathcal{O}$ is **totally positive** if both α and its conjugate α' are positive.

Proposition 5.1 *Fix $a, b > 0$ such that $1 < \frac{b}{a} < \eta^2$, where $\eta > 1$ is a fundamental unit of \mathcal{O} . If $\Pi(x; a, b)$ denotes the number of totally positive primes $\rho \in \mathcal{O}$ satisfying $N(\rho) \leq x$ and $a < \frac{\rho'}{\rho} \leq b$, then*

$$\Pi(x; a, b) \sim \frac{\ln b - \ln a}{2h \ln(\eta^2)} \cdot \frac{x}{\ln x}.$$

Observe that the factor of h and the conditions on a and b ensure that we are looking at specific associates of certain prime elements in \mathcal{O} . To put the presence of the other constant factor in the conclusion of the proposition into some perspective, consider the logarithm map $x \mapsto \ln |x|$. In particular, this transforms $1 < \frac{b}{a} < \eta^2$ to $0 < \ln b - \ln a < \ln(\eta^2)$. By interpreting this as saying that the interval $[\ln a, \ln b]$ is contained in $[0, \ln(\eta^2)]$, the significance of the quantity $\frac{\ln b - \ln a}{\ln(\eta^2)}$ resembles that of $\frac{\theta_2 - \theta_1}{2\pi}$ from our prime number theorem in Proposition 4.1.

Now, we use this result to readily establish the density of quotients of primes in \mathcal{O} .

Theorem 5.2 *The set of quotients of primes in a real quadratic number ring \mathcal{O} is dense in \mathbb{R} .*

Proof: We need to show that any given interval (a, b) contains a quotient of primes in \mathcal{O} . By reducing the length of the interval or using symmetry, we can assume without loss of generality not only that $a, b > 0$, but also $1 < \frac{b}{a} < \eta^2$.

It is easy to check that if $m = \frac{a+b}{2}$, then $1 < \frac{m}{a} < \eta^2$. Now, we apply Proposition 5.1 to the interval (a, m) : for sufficiently large x , there exists a totally positive prime $\pi \in \mathcal{O}$ with norm at most x such that $a < \frac{\pi'}{\pi} \leq m$. Since $m < b$, this implies that $a < \frac{\pi'}{\pi} < b$.

It remains to show that π' is a prime in \mathcal{O} . To this end, suppose that $\pi' \mid \alpha\beta$ for some $\alpha, \beta \in \mathcal{O}$. Taking conjugates yields $\pi \mid \alpha'\beta'$. Since π is a prime, this implies that $\pi \mid \alpha'$ or $\pi \mid \beta'$. By applying conjugation again, we obtain $\pi' \mid \alpha$ or $\pi' \mid \beta$ as required. □

Note: An alternate proof suggested in private conversation with Garcia uses previous quotient set density theorems and bypasses the use of Proposition 5.1. As a trade-off, it does not allow for the possibility of using non-rational prime numbers to establish the density result.

It relies on the following fact: Given a real quadratic number field $\mathbb{Q}(\sqrt{d})$ with $d > 0$ and square-free, an odd rational prime p remains prime in \mathcal{O} iff d is not a square modulo p .

By fixing d , this condition gives a set of congruences in terms of d which p satisfies. Pick one such congruence condition; by Dirichlet's Theorem on Arithmetic Progressions, there are infinitely many such primes. Now, apply the result of Micholson [10] and Starni [13] (as stated before Proposition 2.3) to this congruence condition, and we are done. \square

6 Concluding Remarks

A unifying thread through the proofs of the density theorems presented here is having a suitable version of a prime number theorem around to permit us to establish the desired results. The existence of such theorems is rather remarkable in itself and are key results belonging to both Analytic and Algebraic Number Theory.

The curious reader may wonder whether there exist congruence class analogues for quotients of primes in quadratic number rings other than $\mathbb{Z}[i]$. This is indeed the case at least when these are UFDs, and the reader is encouraged to find and state the appropriate versions of the prime number theorem to establish these results; they can be found in [2], [7], and [11].

As for some open questions, one can try to state and establish analogous density results for other algebraic number rings. For instance, show that quotients of prime elements in the cyclotomic number ring $\mathbb{Z}[e^{2\pi i/n}]$ (for any integer $n > 2$) is dense in \mathbb{C} . Possibly more interestingly, one can consider the noncommutative ring of Hurwitz quaternions

$$H = \left\{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}$$

and try to prove that $\{\pi_1\pi_2^{-1} \mid \pi_1, \pi_2 \in H \text{ are prime}\}$ is dense in \mathbb{H} , the set of quaternions (over \mathbb{R}).

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